Linear Algebra
[KOMS119602] - 2022/2023

12.2 - Types of Linear Transformation

Dewi Sintiari

Computer Science Study Program
Universitas Pendidikan Ganesha

Week 13 (December 2022)

Learning objectives

After this lecture, you should be able to:

1. explain the concept of various types of linear transformation among vectors in vector spaces;
2. perform a linear transformation (reflection, projection, rotation, dilation, expansion, shear) on a vector in a vector space.

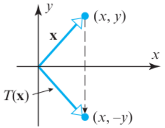
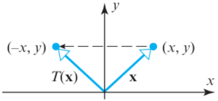
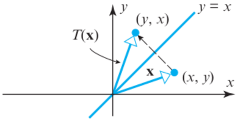
Basic Matrix Transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$

(page 259 of Elementary LA Applications book)

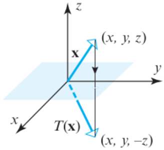
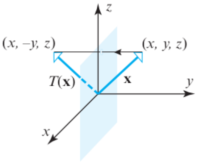
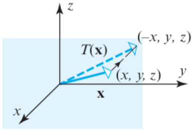
1. Reflection

Reflection operators on $\mathbb{R}^2$

Reflection operators are operators on $\mathbb{R}^2$ (or $\mathbb{R}^3$) that maps each point into its symmetric image about a fixed line or a fixed plane that contains the origin.

<table>
<tr><th>Operator</th><th>Illustration</th><th>Images of $\mathbf{e}_1$ and $\mathbf{e}_2$</th><th>Standard Matrix</th></tr>
<tr><td>Reflection about the x-axis
$T(x, y) = (x, -y)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$
$T(\mathbf{e}_2) = T(0, 1) = (0, -1)$</td><td>$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$</td></tr>
<tr><td>Reflection about the y-axis
$T(x, y) = (-x, y)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$
$T(\mathbf{e}_2) = T(0, 1) = (0, 1)$</td><td>$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>Reflection about the line $y = x$
$T(x, y) = (y, x)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$
$T(\mathbf{e}_2) = T(0, 1) = (1, 0)$</td><td>$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$</td></tr>
</table>

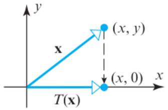
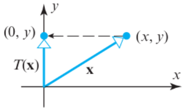
Reflection operators on $\mathbb{R}^3$

Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

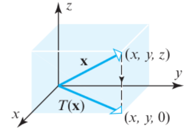
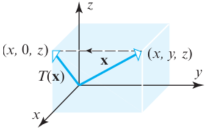
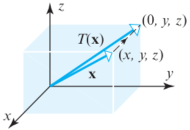
2. Projection

Projection operators on $\mathbb{R}^2$

Projection operators or orthogonal projection operators are matrix operators on $\mathbb{R}^2$ (or $\mathbb{R}^3$) that map each point into its orthogonal projection onto a fixed line or plane through the origin.

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projection operators on $\mathbb{R}^3$

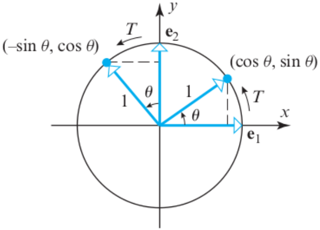
Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Rotation

Rotation operators for $\mathbb{R}^2$

Rotation operators are matrix operators on $\mathbb{R}^2$ or $\mathbb{R}^3$ that move points along arcs of circles centered at the origin.

How to find the standard matrix for the rotation operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that moves points counterclockwise about the origin through a positive angle θ ?



$$T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta) \quad \text{and} \quad T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$$

The standard transformation matrix for T is:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Review on "angle"

Conversion from ^{o} to rad

- $180^o = 1\pi$ rad
- $1^o = \frac{\pi}{180}$ rad

Rotation operators for $\mathbb{R}^2$ (*cont.*)

The matrix:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

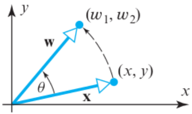
is called the rotation matrix for $\mathbb{R}^2$.

Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $\mathbf{w} = (w_1, w_2)$ be its image under the rotation. Then:

$$\mathbf{w} = R_\theta \mathbf{x}$$

with:

$$w_1 = x\cos\theta - y\sin\theta$$
$$w_2 = x\sin\theta + y\cos\theta$$

Operator	**Illustration**	**Rotation Equations**	**Standard Matrix**
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example: a rotation operator

Find the image of $\mathbf{x} = (1, 1)$ under a rotation of $\pi/6$ rad $(= 30^o)$ about the origin.

Solution:

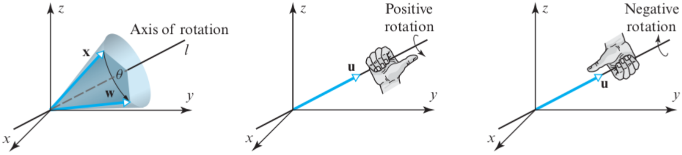
We know that: $\sin(\pi/6) = \frac{1}{2}$ and $\cos(\pi/6) = \frac{\sqrt{3}}{2}$.

By the previous formula:

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

Rotations in $\mathbb{R}^3$

Rotations in $\mathbb{R}^3$ is commonly described as axis of rotation and a unit vector **u** along that line.

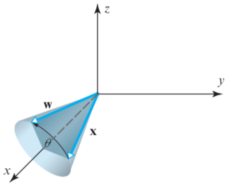
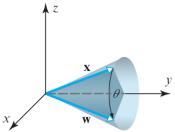
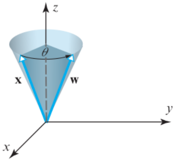


(*a*) Angle of rotation (*b*) Right-hand rule

Right-hand rule is used to establish a sign for the angle for rotation.

- If the axes are the axis x, y, or z, then take the unit vectors **i**, **j**, and **k** respectively.
- An angle of rotation will be *positive if it is counterclockwise* looking toward the origin along the positive coordinate axis and will be *negative if it is clockwise*.

Rotations in $\mathbb{R}^3$

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. Dilation and contraction

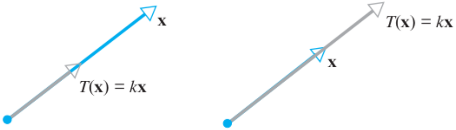
Dilation & contraction

Let $k \in \mathbb{R}, k \geq 0$. The operator:

$$T(\mathbf{x}) = k\mathbf{x}$$

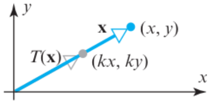
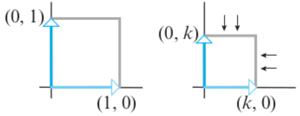
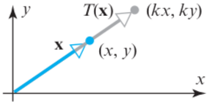
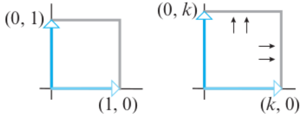
on $\mathbb{R}^2$ or $\mathbb{R}^3$ defines the increment or decrement of the length of vector $\mathbf{x}$ by a factor of k.

- If $k > 1$, it is called a dilation with factor k;
- If $0 \leq k \leq 1$, it is called a contraction with factor k.

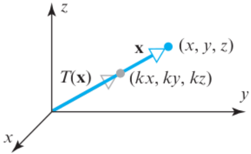
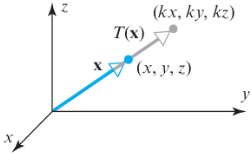


(a) $0 \leq k < 1$ (b) $k > 1$

Dilation & contraction on $\mathbb{R}^2$

Operator	Illustration $T(x, y) = (kx, ky)$	Effect on the Unit Square	Standard Matrix
Contraction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k in R^2 $(k > 1)$			

Dilation & contraction on $\mathbb{R}^3$

Operator	Illustration $T(x, y, z) = (kx, ky, kz)$	Standard Matrix
Contraction with factor k in R^3 $(0 \leq k < 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k in R^3 $(k > 1)$		

5. Expansion and compression

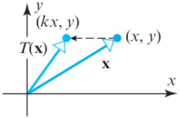
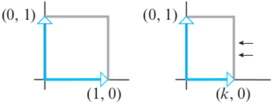
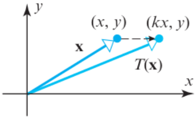
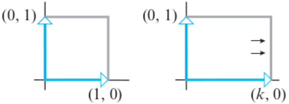
Expansion and compression

In a dilation or contraction of $\mathbb{R}^2$ or $\mathbb{R}^3$, **all coordinates** are multiplied by a non-negative factor k.

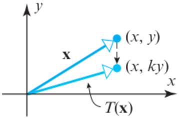
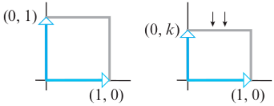
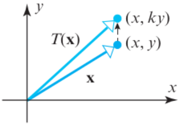
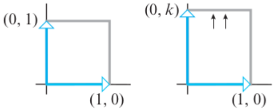
Now what if **only one coordinate** is multiplied by k ?

- If $k > 1$, it is called the expansion with factor k in the direction of a coordinate axis (x, y, or z);
- If $0 \leq k \leq 1$, it is called compression

Expansion and compression in $\mathbb{R}^2$ (in x-direction)

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Unit Square	Standard Matrix
Compression in the x-direction with factor k in R^2 $(0 \leq k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion in the x-direction with factor k in R^2 $(k > 1)$			

Expansion and compression in $\mathbb{R}^2$ (in y-direction)

Operator	Illustration $T(x, y) = (x, ky)$	Effect on the Unit Square	Standard Matrix
Compression in the y-direction with factor k in R^2 $(0 \leq k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

6. Shear

Shear

A matrix operator of the form:

$$T(x, y) = (x + ky, y)$$

translates a point (x, y) in the xy-plane parallel to the x-axis by an amount ky that is proportional to the y-coordinate of the point.

This is called shear in the x-direction by a factor k.

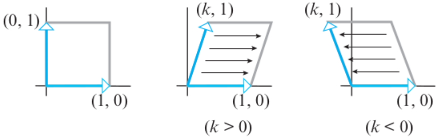
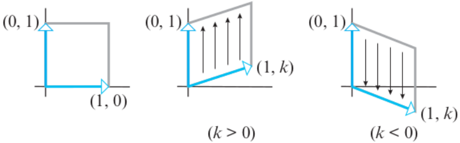
Similarly, a matrix operator:

$$T(x, y) = (x, y + kx)$$

is called shear in the y-direction by a factor k.

When $k > 0$, then the shear is in the positive direction. When $k < 0$, it is in the negative direction.

Shear

Operator	Effect on the Unit Square	Standard Matrix
Shear in the x-direction by a factor k in R^2 $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction by a factor k in R^2 $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Example

Describe the matrix operator whose standard matrix is as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

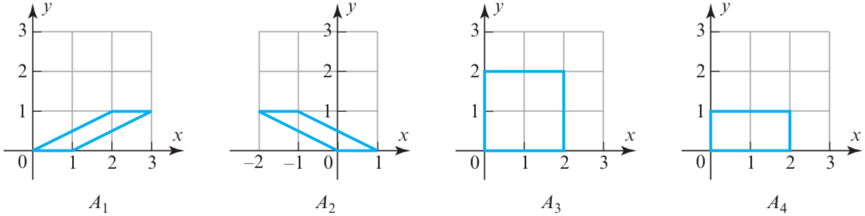
Solution:

From the tables on the previous slides, we can see that:

- A_1 corresponds to a shear in the x-direction by a factor 2;
- A_2 corresponds to a shear in the x-direction by a factor -2;
- A_3 corresponds to a dilation with factor 2;
- A_4 corresponds to an expansion in the x-direction with factor 2.

Example (*cont.*)

Describe geometrically the result of the transformation:



Exercise